

## Flavors in the microscopic approach to $\mathcal{N} = 1$ gauge theories

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# Flavors in the microscopic approach to $\mathcal{N} = 1$ gauge theories

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**ABSTRACT:** In this note, we solve an extended version of the  $\mathcal{N} = 1$  super Yang-Mills theory with gauge group  $U(N)$ , an adjoint chiral multiplet and  $N_f$  flavors of quarks, by using the  $\mathcal{N} = 1$  microscopic formalism based on Nekrasov's sums over colored partitions. Our main new result is the computation of the general mesonic operators. We prove that the generalized Konishi anomaly equations with flavors are satisfied at the non-perturbative level. This yields in particular a microscopic, first principle derivation of the matrix model disk diagram contributions that must be included in the Dijkgraaf-Vafa approach.

**KEYWORDS:** Supersymmetric gauge theory, Duality in Gauge Field Theories, Gauge-gravity correspondence, Nonperturbative Effects

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**1 Introduction**

The recently developed microscopic formalism to  $\mathcal{N} = 1$  supersymmetric gauge theory [1–3] is a first-principle approach that allows in principle to solve rigorously a general  $\mathcal{N} = 1$  gauge theory in the chiral sector. It is based on the direct calculation of the relevant path integrals and relies heavily on Nekrasov’s instanton technology [4], suitably adapted to the  $\mathcal{N} = 1$  context. The goal of the present paper is to develop the formalism in the case where fundamental quark flavors are present. Including quarks brings several new interesting features that we shall explain in details in the following.

The model we consider is the  $U(N)$  supersymmetric gauge theory with an adjoint chiral multiplet  $X$  and chiral multiplets  $Q_f$  and  $\tilde{Q}^f$ ,  $1 \leq f \leq N_f$ , in the fundamental and anti-fundamental representations respectively. We shall restrict ourselves to the cases  $N_f \leq 2N$ .

The lagrangian is given by

$$\begin{aligned} \mathcal{L} = \text{Re} \int d^2\theta \frac{\tau}{4\pi i} \text{Tr} W^\alpha W_\alpha + \int d^2\theta d^2\bar{\theta} \text{Tr} X^\dagger e^{2V} X \\ + \int d^2\theta d^2\bar{\theta} Q^{\dagger f} e^{2V} Q_f + \int d^2\theta d^2\bar{\theta} \tilde{Q}_f^\dagger e^{2V} \tilde{Q}^f + 2N \text{Re} \int d^2\theta \mathcal{W}, \end{aligned} \tag{1.1}$$

where

$$\tau = \frac{\vartheta}{2\pi} + i \frac{4\pi}{g^2} \tag{1.2}$$

is the gauge coupling constant and

$$\mathcal{W} = \frac{-1}{16\pi^2} \text{Tr} (V(X)W^\alpha W_\alpha) + \text{Tr} W(X) + {}^T \tilde{Q}^f m^f_{\tilde{f}}(X) Q_f. \tag{1.3}$$

The polynomials  $V(z)$ ,  $W(z)$  and  $m_{\tilde{f}}^f(z)$  are parametrized as follows:

$$NV(z) = \sum_{k \geq 2} k t_k z^{k-1}, \tag{1.4}$$

$$W(z) = \sum_{k \geq 0} \frac{g_k}{k+1} z^{k+1}, \tag{1.5}$$

$$m_{\tilde{f}}^f(z) = \sum_{k \geq 0} m_{k,\tilde{f}}^f z^k. \tag{1.6}$$

It is useful to introduce

$$U(z) = \det(m(z)) = U_0 \prod_{Q=1}^{\ell} (z - b_Q) \tag{1.7}$$

and

$$t(z) = \sum_{k \geq 1} \frac{t_k}{k+1} z^{k+1}, \tag{1.8}$$

with

$$t_1 = \ln(U_0 q). \tag{1.9}$$

The coupling  $q$  is the instanton factor in the model. In the asymptotically free case ( $N_f < 2N$ ) it is expressed in terms of the dynamically generated scale  $\Lambda$  as

$$q = \Lambda^{2N - N_f}, \tag{1.10}$$

whereas in the case of vanishing  $\beta$  function ( $N_f = 2N$ ) we have

$$q = e^{2\pi i \tau} = e^{i\vartheta - 8\pi^2/g^2}. \tag{1.11}$$

The set of couplings  $\{t_k\}_{k \geq 2}$ ,  $\{g_k\}$ ,  $\{m_{k,\tilde{f}}^f\}$  and  $\{b_Q\}$  will be denoted by  $\mathbf{t}$ ,  $\mathbf{g}$ ,  $\mathbf{m}$  and  $\mathbf{b}$  respectively. In the most standard case,  $t_k = 0$  for  $k \geq 2$ , but as explained in [3] it is actually very natural and convenient to consider the more general theory with arbitrary couplings  $\mathbf{t}$ .

The chiral ring of the model is generated by the operators

$$\text{Tr } X^k, \quad \text{Tr } W^\alpha W_\alpha X^k, \quad {}^T \tilde{Q}^{\tilde{f}} X^k Q_f \tag{1.12}$$

and our goal is to compute, from first principles, the corresponding expectation values. These expectation values are multi-valued analytic functions of the parameters  $\mathbf{t}$ ,  $\mathbf{g}$ ,  $\mathbf{m}$  and  $q$ . The multi-valuedness comes from the existence of several distinct vacua in the theory. At the classical level, the most general supersymmetric vacuum, which is obtained by extremizing the tree-level superpotential, can be labeled as  $|N_i; \nu_Q\rangle_{\text{cl}}$ , where the  $N_i \geq 0$  and  $\nu_Q \in \{0, 1\}$  are integers satisfying the constraint

$$\sum_{i=1}^{\text{deg } W'} N_i + \sum_{Q=1}^{\ell} \nu_Q = N. \tag{1.13}$$

The  $N_i$ s and  $\nu_Q$ s denote the number of eigenvalues of the matrix  $X$  that are equal classically to the  $i^{\text{th}}$  root of  $W'(z)$  and to  $b_Q$ , respectively. The  $U(N)$  gauge symmetry in a vacuum  $|N_i; \nu_Q\rangle_{\text{cl}}$  is thus broken down to a product of  $U(N_i)$  factors. The number of non-trivial factors of the unbroken gauge group, i.e. the number of non-zero  $N_i$ s, is called the rank of the vacuum. In the quantum theory, chiral symmetry breaking yields a larger degeneracy of the vacua, that are then labeled as  $|N_i, k_i; \nu_Q\rangle$  where  $0 \leq k_i \leq N_i - 1$ . This structure follows from the extremization of the microscopic quantum superpotential that will be introduced in the next section.

The expectation values of the operators (1.12) are most conveniently encoded in the generating functions

$$\mathcal{R}(z) = \sum_{k \geq 0} \frac{\langle \text{Tr } X^k \rangle}{z^{k+1}}, \tag{1.14}$$

$$\mathcal{S}(z) = -\frac{1}{16\pi^2} \sum_{k \geq 0} \frac{\langle \text{Tr } W^\alpha W_\alpha X^k \rangle}{z^{k+1}}, \tag{1.15}$$

$$\mathcal{G}_{\tilde{f}}^{\tilde{f}}(z) = \sum_{k \geq 0} \frac{\langle \text{Tr } \tilde{Q}^{\tilde{f}} X^k Q_{\tilde{f}} \rangle}{z^{k+1}}. \tag{1.16}$$

One of our main result is to show that these generating functions satisfy the following set of algebraic equations,

$$NW'(z)\mathcal{S}(z) - \mathcal{S}(z)^2 = \Delta_{\mathcal{S}}(z), \tag{1.17}$$

$$Nm_{\tilde{f}}^{f'}(z)\mathcal{G}_{\tilde{f}}^{\tilde{f}}(z) - \mathcal{S}(z)\delta_{\tilde{f}}^{f'} = \Delta_{\tilde{f}}^{f'}(z), \tag{1.18}$$

$$N\mathcal{G}_{\tilde{f}}^{\tilde{f}'}(z)m_{\tilde{f}}^{\tilde{f}}(z) - \mathcal{S}(z)\delta_{\tilde{f}}^{\tilde{f}'} = \tilde{\Delta}_{\tilde{f}}^{\tilde{f}'}(z), \tag{1.19}$$

$$t'''(z)\mathcal{S}(z) + NW'(z)\mathcal{R}(z) + Nm_{\tilde{f}}^{\tilde{f}}(z)\mathcal{G}_{\tilde{f}}^{\tilde{f}}(z) - 2\mathcal{S}(z)\mathcal{R}(z) = \Delta_{\mathcal{R}}(z), \tag{1.20}$$

where  $\Delta_{\mathcal{S}}$ ,  $\Delta_{\tilde{f}}^{f'}$ ,  $\tilde{\Delta}_{\tilde{f}}^{\tilde{f}'}$  and  $\Delta_{\mathcal{R}}$  are polynomials. These equations are the famous generalized Konishi anomaly equations [5], adapted to the case where flavors are included in the model [6] and suitably generalized to the extended theory corresponding to having arbitrary couplings  $\mathbf{t}$ . These equations are at the heart of the Dijkgraaf-Vafa matrix model formalism [7, 8], where they follow directly from the planar loop equations of the matrix model. They were understood at the perturbative level (i.e. in a fixed classical background gauge field) in [5, 6], but a full non-perturbative proof requires much more work as explained in great details in [2, 3]. This is where the microscopic formalism shows its full power. It is remarkable to reproduce the planar matrix model result (which, when flavors are present, also include disk diagrams [8]) from *finite*  $N$  gauge theory path integral calculations. As we shall see, these integrals can be reduced to non-trivial sums over colored partitions.

The constraints (1.17), (1.18), (1.19) and (1.20) do not fix completely the expectation values. There remains undetermined coefficients in the polynomials that appear in the right hand side of these equations. This ambiguity is completely removed by the fact that the number of colors  $N$  in the gauge theory is finite and thus only a finite number of the

operators (1.12) are algebraically independent [9–11]. Mathematically, this is translated into quantization conditions for the periods of  $\mathcal{R} dz$ ,

$$\oint \mathcal{R}(z) dz \in 2\pi i \mathbb{Z}. \tag{1.21}$$

In our microscopic formalism, this is satisfied by construction, since the operators are built explicitly from finite  $N \times N$  matrices and the relations (1.21) will be easy to check.<sup>1</sup>

The plan of the paper is as follows. In section 2, we explain the microscopic formalism in the case of the model (1.1). In sections 3, 4 and 5 we compute the scalar, glueball and meson operators respectively. Finally, in section 6, we extremize the microscopic superpotential, derive the anomaly equations and discuss some general features of the solution. We have also included an appendix containing some technicalities used in the main text.

## 2 The microscopic formalism

The starting point of the microscopic formalism [1] is to consider the expectation values of the operators (1.12) *with fixed boundary conditions at infinity* for the adjoint scalar field  $X$ ,

$$X_\infty = \text{diag} (a_1, \dots, a_N). \tag{2.1}$$

The eigenvalues at infinity

$$\mathbf{a} = (a_1, \dots, a_N) \tag{2.2}$$

are arbitrary fixed complex numbers. We could also try to impose arbitrary boundary conditions at infinity for the quark fields  $Q_f$  and  $\tilde{Q}^{\tilde{f}}$ , but we prefer in the present paper to first integrate over these fields *exactly* in the path integrals. The model is then reduced to the case with no flavor, but with extra determinant-like factors due to the integration over the quarks. The expectation value of an arbitrary chiral operator  $\mathcal{O}$  with the boundary conditions (2.1) will be denoted by  $\langle \mathbf{a} | \mathcal{O} | \mathbf{a} \rangle$  and the corresponding generating functions in our model are given by

$$R(z; \mathbf{a}) = \sum_{k \geq 0} \frac{\langle \mathbf{a} | \text{Tr} X^k | \mathbf{a} \rangle}{z^{k+1}}, \tag{2.3}$$

$$S(z; \mathbf{a}) = -\frac{1}{16\pi^2} \sum_{k \geq 0} \frac{\langle \mathbf{a} | \text{Tr} W^\alpha W_\alpha X^k | \mathbf{a} \rangle}{z^{k+1}}, \tag{2.4}$$

$$G_f^{\tilde{f}}(z; \mathbf{a}) = \sum_{k \geq 0} \frac{\langle \mathbf{a} | \text{Tr} \tilde{Q}^{\tilde{f}} X^k Q_f | \mathbf{a} \rangle}{z^{k+1}}. \tag{2.5}$$

Clearly, the expectation values  $\langle \mathbf{a} | \mathcal{O} | \mathbf{a} \rangle$  are not physical and the corresponding generating functions (2.3), (2.4) and (2.5) do not coincide with the physical generating functions (1.14), (1.15) and (1.16).<sup>2</sup> In particular, the functions  $R$ ,  $S$  and  $G_f^{\tilde{f}}$  depend on the

<sup>1</sup>In the Dijkgraaf-Vafa matrix model formalism, on the other hand, the conditions follow from the extremization of the postulated glueball superpotential with all the required Veneziano-Yankielowicz terms included [12] and are thus highly non-trivial.

<sup>2</sup>We have changed slightly our notation with respect to references [1–3];  $R$  and  $S$  in the present paper correspond to  $R_{\text{mic}}$  and  $S_{\text{mic}}$  in our previous works.

arbitrary boundary conditions  $\mathbf{a}$  (as well of course on the other parameters in the model), whereas the physical correlators depend on a choice of vacuum but not on  $\mathbf{a}$ .

The interest in considering the correlators for fixed boundary conditions at infinity is that, at least in an open set in  $\mathbf{a}$ -space, they can always be computed by summing a convergent instanton series [1]. The functions at arbitrary  $\mathbf{a}$  (outside the radius of convergence of the instanton series) are then obtained uniquely by analytic continuations.

There exists a quantum superpotential  $W_{\text{mic}}(\mathbf{a})$  for the boundary conditions  $\mathbf{a}$  [1]. One of the fundamental property of this quantum superpotential is that the solutions  $\mathbf{a} = \mathbf{a}^*$  of the equations

$$\frac{\partial W_{\text{mic}}}{\partial a_i}(\mathbf{a} = \mathbf{a}^*) = 0 \quad (2.6)$$

are in one-to-one correspondence with the full set of quantum vacua of the theory [1]. The physical correlators in a given vacuum are then obtained by plugging the corresponding solution  $\mathbf{a} = \mathbf{a}^*$  to (2.6) into the generating functions (2.3), (2.4) and (2.5),

$$\mathcal{R}(z) = R(z; \mathbf{a}^*), \quad (2.7)$$

$$\mathcal{S}(z) = S(z; \mathbf{a}^*), \quad (2.8)$$

$$\mathcal{G}_{\tilde{f}}^{\tilde{f}}(z) = G_{\tilde{f}}^{\tilde{f}}(z; \mathbf{a}^*). \quad (2.9)$$

The superpotential  $W_{\text{mic}}$  is always unambiguously determined in terms of the expectation values by a  $U(1)_R$  symmetry of the model. In our case, the charges for the relevant  $U(1)_R$  are given by

$$\begin{array}{cccccccccccc} \theta & W^\alpha & X & Q_f & \tilde{Q}^{\tilde{f}} & \mathbf{a} & \mathbf{g} & \mathbf{m} & U_0 & \mathbf{b} & \mathbf{t} & \mathbf{q} & \mathcal{W} \\ U(1)_R & 1 & 1 & 0 & 1 & 1 & 0 & 2 & 0 & 0 & 0 & 0 & 2 \end{array}, \quad (2.10)$$

where  $\theta$  is a superspace coordinate and  $\mathcal{W}$  an arbitrary superpotential. The Ward identity associated with this  $U(1)_R$  reads

$$W_{\text{mic}}(\mathbf{a}) = \sum_{k \geq 0} g_k \frac{\partial W_{\text{mic}}}{\partial g_k}. \quad (2.11)$$

Using the standard supersymmetric Ward identity

$$\langle \mathbf{a} | \text{Tr } X^{k+1} | \mathbf{a} \rangle = (k+1) \frac{\partial W_{\text{mic}}}{\partial g_k}, \quad (2.12)$$

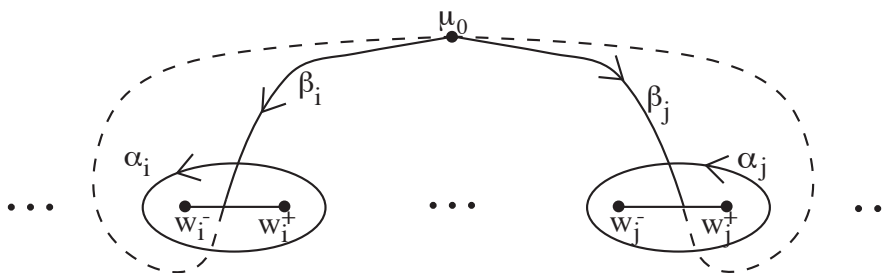
we thus get the fundamental formula

$$W_{\text{mic}}(\mathbf{a}) = \langle \mathbf{a} | \text{Tr } W(X) | \mathbf{a} \rangle \quad (2.13)$$

relating the quantum superpotential to the correlators of the chiral operators  $\text{Tr } X^k$ . We shall also need two additional supersymmetric Ward identities, similar to (2.12), that read

$$-\frac{1}{16\pi^2} \langle \mathbf{a} | \text{Tr } W^\alpha W_\alpha X^k | \mathbf{a} \rangle = \frac{N}{k+1} \frac{\partial W_{\text{mic}}}{\partial t_{k+1}}, \quad (2.14)$$

$$\langle \mathbf{a} | \text{Tr } \tilde{Q}^{\tilde{f}} X^k Q_f | \mathbf{a} \rangle = \frac{\partial W_{\text{mic}}}{\partial m_{k,\tilde{f}}^f}. \quad (2.15)$$



**Figure 1.** The hyperelliptic Riemann surface  $\mathcal{C}$ , with the cycles  $\alpha_i$  and chains  $\beta_i$  used in the main text. Note that the contours  $\alpha_i$  are chosen such that they do not encircle the points  $b_Q$  (this is needed in section 5). The point  $\mu_0$  is taken to infinity.

### 3 The shape function and the scalar operators

All the chiral correlators for given boundary conditions  $\mathbf{a}$  can be computed using Nekrasov’s instanton technology. We plan to provide a general discussion of this technology in a purely  $\mathcal{N} = 1$  context in a subsequent paper, but for our present purposes all the relevant formulas can be obtained from the existing literature [4, 13, 14] through rather simple generalizations.

It is convenient to present first the solution when all the parameters in the problem are real, the  $a_i$ s being widely separated and the  $b_Q$ s sufficiently smaller than all the  $a_i$ s. In this case all the expectation values can be expressed in terms of a so-called shape function  $f$ ,

$$\langle \mathbf{a} | \mathcal{O} | \mathbf{a} \rangle = \mathcal{O}[f]. \tag{3.1}$$

For example,

$$R(z; \mathbf{a}) = \frac{1}{2} \int_{\mathbb{R}} dy \frac{f''(y)}{z - y}. \tag{3.2}$$

The shape function  $f$  extremizes the functional

$$\begin{aligned} \mathcal{F} = & -\frac{1}{8} \int dx dy (x - y)^2 (\ln |x - y| - 3/2) f''(x) f''(y) \\ & + \frac{1}{4} \int dx \sum_{Q=1}^{\ell} (x - b_Q)^2 (\ln(x - b_Q) - 3/2) f''(x) + \frac{1}{2} \int dx t(x) f''(x) \end{aligned} \tag{3.3}$$

with the constraints

$$\text{support}[f''] = \bigcup_{i=1}^N I_i, \tag{3.4}$$

$$\int_{I_i} dx f''(x) = 2, \tag{3.5}$$

$$\int_{I_i} dx x f''(x) = 2a_i, \tag{3.6}$$

$$f(x) = \sum_{i=1}^N |x - a_i|, \quad \forall x \notin \bigcup_{i=1}^N I_i, \tag{3.7}$$



where the  $N$  intervals

$$I_i = [w_i^-, w_i^+] \quad (3.8)$$

are disjoint. The shape function plays a rôle that is very similar to the density of eigenvalues in planar matrix models. In the present context, the integrals over the instanton moduli space are reduced by localization to sums over particular field configurations labeled by colored partitions, and the shape function describes the dominating colored partition in the limit of vanishing  $\Omega$ -background [13]. A few more details are given in the appendix A.2. The correlator  $\langle \mathbf{a} | \mathcal{O} | \mathbf{a} \rangle$  for arbitrary complex values of the parameters are obtained by analytic continuation, as will be clear in the following.

The formula (3.3) is a simple generalization of cases that were previously studied in the literature. The flavorless case of our model was studied for  $\mathbf{g} = 0$  in [14] and for arbitrary  $\mathbf{g}$  in [3]. The  $\mathbf{t} = \mathbf{g} = 0$  theory was studied in [4, 13] in the case where the mass matrix  $m_{\tilde{f}}^f(z)$  is a linear function of  $z$ .

For our purposes, we need to solve the extremization problem in the general case where all the couplings are turned on simultaneously. The equation  $\delta\mathcal{F}/\delta f(x) = 0$  reads in this case

$$\int dy \ln|x-y| f''(y) = \sum_{Q=1}^{\ell} \ln(x-b_Q) + t''(x), \quad \forall x \in \bigcup_{i=1}^N I_i. \quad (3.9)$$

Instead of working with  $f$ , it is more convenient to study  $R$ . Let us note that (3.2) implies that  $f$  can be obtained from the discontinuity of  $R$  across the intervals  $I_i$ ,

$$R(x+i0) - R(x-i0) = -i\pi f''(x). \quad (3.10)$$

The first derivative of (3.9) yields

$$R(x+i0) + R(x-i0) = \sum_{Q=1}^{\ell} \frac{1}{x-b_Q} + t'''(x) \quad (3.11)$$

$$= \frac{U'(x)}{U(x)} + t'''(x), \quad \forall x \in \bigcup_{i=1}^N I_i. \quad (3.12)$$

This constraint implies that  $R$  is a well-defined meromorphic function on the hyperelliptic curve

$$\mathcal{C} : y^2 = \prod_{i=1}^N (z - w_i^+)(z - w_i^-). \quad (3.13)$$

This curve, with some useful contours, is depicted in figure 1. In particular,  $R(z)$  is a two-valued function. From now on, we shall denote by  $R(z)$  its value on the first sheet, where the asymptotic condition at infinity

$$R(z) = \frac{N}{z} + \mathcal{O}(1/z^2) \quad (3.14)$$

is valid, and by  $\hat{R}(z)$  its value on the second sheet. More generally, a hat on a function defined on (3.13) will always mean that we consider its value on the second sheet. For example,  $\hat{y} = -y$ . Equation (3.12) is equivalent to

$$R(z) + \hat{R}(z) = \frac{U'(z)}{U(z)} + t'''(z). \quad (3.15)$$

As any other meromorphic function on the curve  $\mathcal{C}$ ,  $R$  can be written in the form

$$R(z) = r_1(z) + \frac{r_2(z)}{y}, \quad (3.16)$$

where  $r_1$  and  $r_2$  are rational functions. Equation (3.15) implies that

$$r_1(z) = \frac{1}{2} \left( \frac{U'(z)}{U(z)} + t'''(z) \right). \quad (3.17)$$

Moreover, from the integral representation (3.2) and the constraints on the shape function  $f$ ,  $R$  cannot have poles on the first sheet. From (3.15), we deduce that the only poles of  $R$  are on the second sheet at  $z = b_Q$  with residue one. This implies that

$$r_2(z) = \frac{p(z)}{U(z)}, \quad (3.18)$$

for some polynomial  $p(z)$ . The asymptotic behaviour (3.14) implies that  $\deg p = \deg t''' + N + \ell$  and imposes  $\deg t''' + 2$  constraints on the coefficients of  $p$ . Matching the residues at the poles on the second sheet implies  $\ell$  additional constraints

$$p(b_Q) = -\frac{1}{2} U'(b_Q) y(b_Q). \quad (3.19)$$

There remains  $N - 1$  unknown coefficients in  $p$ , as well as the  $2N$  unknown branching points  $w_i^\pm$  on the curve (3.13). These  $3N - 1$  parameters are fixed by the following  $3N - 1$  independent constraints on the periods of  $R dz$ ,

$$\oint_{\alpha_i} R(z) dz = 2i\pi, \quad (3.20)$$

$$\oint_{\alpha_i} zR(z) dz = 2i\pi a_i, \quad (3.21)$$

$$\int_{\beta_i} R(z) dz = \ln \left( \frac{U_0 q}{\mu_0^{2N-\ell}} \right) + NV(\mu_0) + 2i\pi\mathbb{Z}. \quad (3.22)$$

The contours  $\alpha_i$  and  $\beta_i$  are depicted in figure 1. The cut-off  $\mu_0$  is always understood to be taken to infinity at the end of the calculations. Let us note that of all the constraints that determine  $R$ , none depends on  $\mathbf{g}$  and thus  $R$  itself will not depend on  $\mathbf{g}$  (but will of course depend on  $\mathbf{a}$ ,  $\mathbf{t}$  and  $\mathbf{b}$ ). Note that the physical generating function  $\mathcal{R}$  will depend non-trivially on  $\mathbf{g}$  through the solutions  $\mathbf{a}^*$  of (2.6).

Equations (3.20) and (3.21) directly follow from integrating (3.10) and  $x$  times (3.10) over the intervals  $I_i$ s and then using (3.5) and (3.6). Equation (3.22) is more interesting.

It comes from the *integrated* form (3.9) of the variational equation for the shape function. To see this, let us introduce

$$\phi(z) = \int_{\mu_0}^z R(z') dz' + N \ln \mu_0. \quad (3.23)$$

Since we do not specify the contour used to go from the point at infinity on the first sheet  $\mu_0$  to  $z$  in (3.23), (3.20) shows that  $\phi(z)$  is defined modulo  $2i\pi$  on the first sheet. What happens if we cross one of the branch cuts? Since  $\phi' = R$ , we can integrate (3.12) to find the discontinuity of  $\phi$ ,

$$\phi(x + i0) + \phi(x - i0) = \sum_{Q=1}^{\ell} \ln(x - b_Q) + t''(x) + c_i, \quad \forall x \in I_i. \quad (3.24)$$

A priori, the integration constants  $c_i$  could depend on the cut  $I_i$ . However, by comparing with (3.9), we find that all the  $c_i$ s are actually zero. This means that, by crossing any of the cuts, we go to the same sheet of the function  $\phi$ . In other words, modulo  $2i\pi$ ,  $\phi$  is well-defined on the curve (3.13). In particular,

$$\phi(z) + \hat{\phi}(z) = \sum_{Q=1}^{\ell} \ln(z - b_Q) + t''(z) + 2i\pi\mathbb{Z}. \quad (3.25)$$

This yields

$$\int_{\beta_i} R(z) dz = \hat{\phi}(\mu_0) - \phi(\mu_0) + 2i\pi\mathbb{Z} \quad (3.26)$$

$$= -(2N - \ell) \ln \mu_0 + t''(\mu_0) + 2i\pi\mathbb{Z}, \quad (3.27)$$

which is equivalent to (3.22) thanks to the relation

$$t''(z) = NV(z) + \ln(U_0q), \quad (3.28)$$

see (1.4) and (1.8).

Equations (3.20) and (3.22) imply that

$$\oint R(z) dz \in 2i\pi\mathbb{Z}, \quad (3.29)$$

where the integral is computed along any closed contour on the curve  $\mathcal{C}$  (3.13). In particular, the function

$$F(z) = e^{\phi(z)} \quad (3.30)$$

is well-defined on  $\mathcal{C}$ . Its value on the second sheet is determined by (3.25) to be

$$\hat{F}(z) = e^{\hat{\phi}(z)} = \frac{\prod_{Q=1}^{\ell} (z - b_Q)}{F(z)} e^{t''(z)} = \frac{qU(z)}{F(z)} e^{NV(z)}. \quad (3.31)$$

The function  $F$  has an essential singularity at infinity on the second sheet for non-zero  $V$ . In the special case  $V(z) = 0$ , which corresponds to the conventional theory with

standard gauge kinetic term, this singularity becomes power-like and  $F$  is a meromorphic function on  $\mathcal{C}$ . The solution can then be described more explicitly. For example, for  $\ell \leq 2N$ , (3.31) implies that

$$F(z) + \frac{qU(z)}{F(z)} = H_N(z), \tag{3.32}$$

where  $H_N(z) = (1 + qU_0\delta_{\ell,2N})z^N + \dots$  is a degree  $N$  polynomial. Equivalently,

$$F(z) = \frac{1}{2} \left( H_N(z) + \sqrt{H_N(z)^2 - 4qU(z)} \right). \tag{3.33}$$

Comparing with (3.13), we can relate  $H_N$  to the branching points  $w_i^\pm$ ,

$$H_N(z)^2 - 4qU(z) = (1 - qU_0\delta_{\ell,2N})^2 \prod_{i=1}^N (z - w_i^+)(z - w_i^-). \tag{3.34}$$

The generating function  $R$  then takes the form,

$$R(z) = \frac{F'(z)}{F(z)} = \frac{1}{2} \frac{U'(z)}{U(z)} + \frac{1}{\sqrt{H_N(z)^2 - 4qU(z)}} \left( H'_N(z) - \frac{U'(z)H_N(z)}{2U(z)} \right). \tag{3.35}$$

To finish this section, let us comment on the analytic structure of the solution. The structure that we have described above is valid for generic values of the parameters, but interesting phenomena occur when the boundary eigenvalues  $a_i$  of  $X$  are chosen to coincide with the parameters  $b_Q$ . By carefully analysing our solution, it is not too difficult to show that when  $b_Q$  approaches  $a_i$ , the cut  $I_i = [w_i^-, w_i^+]$  closes. At  $a_i = b_Q$ , the curve  $\mathcal{C}$  degenerates to a genus  $N - 2$  curve and the pole at  $z = b_Q$  is on the first sheet. More generally, if  $p$  distinct  $a_i$ s are equal to  $p$  distinct  $b_Q$ s, the curve degenerates down to genus  $N - 1 - p$  and the generating function  $R$  then has  $p$  poles on the first sheet and  $N - p$  poles on the second sheet. We have illustrated this mechanism on a very simple example in the appendix A.1. The cases  $a_i = b_Q$  can actually be treated directly and most easily at the level of the sums over colored partitions. This is explained in the appendix A.2.

#### 4 The glueball operators

The inclusion of flavors modifies only slightly the computation of the generating function  $S(z; \mathbf{a})$  and thus we can follow closely [3]. The fundamental formula relates  $S''(z)$  to  $R(z)$ ,

$$S''(z) = N \sum_{k \geq 1} g_k \frac{\partial R(z)}{\partial t_k}. \tag{4.1}$$

This is the same as equation (3.21) in [3] and the derivation given in that reference applies without change when flavors are included. We can also follow closely [3] to derive the consequences of (4.1). The only potential difference in the analysis could come from the fact that  $R$  has poles. However, the residue of these poles are  $\mathbf{t}$ -independent and thus they do not enter in (4.1). From [3] we thus know that (4.1) implies that  $S'(z)$  must be a meromorphic function on the curve (3.13) of the form

$$S'(z) = \frac{N}{2} W''(z) + \frac{s(z)}{y}, \tag{4.2}$$

for a certain polynomial  $s$  of degree  $\deg W'' + N$ . The asymptotic condition at infinity on the first sheet

$$S'(z) = \mathcal{O}(1/z^2) \tag{4.3}$$

yields  $\deg W'' + 2$  conditions on  $s$ . Moreover, one must have

$$\oint_{\alpha_i} S'(z) dz = 0 \tag{4.4}$$

which yields  $N - 1$  new independent constraints that determine completely  $s$  and thus  $S$ .

## 5 The generalized meson operators

We are now going to show that the generating function for the generalized meson operators (2.5) is given in terms of the generating function for the glueball operators that we have just computed by the formula

$$G_f^{\tilde{f}}(z) = \frac{1}{N} S(z) (m(z)^{-1})^{\tilde{f}}_f - \frac{1}{N} \sum_{Q=1}^{\ell} \frac{S(b_Q)}{z - b_Q} \text{res}_{w=b_Q} (m^{-1}(w))^{\tilde{f}}_f, \tag{5.1}$$

where  $m(z)$  is the mass matrix polynomial (1.6). Note that all the poles of  $G_f^{\tilde{f}}(z)$  are on the second sheet.

To do the calculation, it is very convenient to use the variations of the functional  $\mathcal{F}$  defined in (3.3). Since  $\mathcal{F}$  is stationary with respect to the changes of the shape function  $f$ , we have the simple formula

$$\delta \mathcal{F} = \frac{1}{2i\pi} \oint_{\alpha} \delta \psi(z) R(z) dz, \tag{5.2}$$

where

$$\psi(z) = \frac{1}{2} \sum_{Q=1}^{\ell} (z - b_Q)^2 (\ln(z - b_Q) - 3/2) + t(z). \tag{5.3}$$

To derive (5.2), we have used (3.10) and we have defined  $\alpha$  to be the sum of the contours that circle around the branch cuts of the curve (3.13),

$$\alpha = \sum_{i=1}^N \alpha_i. \tag{5.4}$$

In particular, using (3.21) and (5.2) for  $\delta = \partial/\partial t_k$ , the quantum superpotential (2.13) can be rewritten

$$W_{\text{mic}} = \frac{1}{2i\pi} \sum_{k \geq 0} \frac{g_k}{k+1} \oint_{\alpha} z^{k+1} R(z) dz = g_0 \sum_{i=1}^N a_i + \sum_{k \geq 1} g_k \frac{\partial \mathcal{F}}{\partial t_k}. \tag{5.5}$$

We now use the relation (2.15) combined with (5.5) in the definition (2.5) to obtain

$$G_f^{\tilde{f}}(z) = \sum_{k \geq 0} \sum_{k' \geq 1} \frac{g_{k'}}{z^{k+1}} \frac{\partial^2 \mathcal{F}}{\partial t_{k'} \partial m_{k, \tilde{f}}^f}. \tag{5.6}$$

The partial derivative of  $\mathcal{F}$  with respect to  $m_{k, \tilde{f}}^f$  is then evaluated using (5.2). This yields

$$G_{f, \tilde{f}}^{\tilde{f}}(z) = \sum_{k \geq 0} \sum_{k' \geq 1} \frac{g_{k'}}{z^{k+1}} \frac{\partial}{\partial t_{k'}} \oint_{\alpha} \frac{dw}{2i\pi} \frac{\partial \psi(w)}{\partial m_{k, \tilde{f}}^f} R(w). \quad (5.7)$$

The function  $\psi$  defined in (5.3) depends on the  $m_{k, \tilde{f}}^f$  only through the  $b_Q$ s and  $t_1$ , see (1.9). It is clear that  $\partial \psi(w) / \partial m_{k, \tilde{f}}^f$  does not depend on  $t_{k'}$  and thus the partial derivative with respect to  $t_{k'}$  in (5.7) acts only on  $R(w)$ . We can then rearrange nicely the formula using (4.1),

$$G_{f, \tilde{f}}^{\tilde{f}}(z) = \frac{1}{N} \sum_{k \geq 0} \frac{1}{z^{k+1}} \oint_{\alpha} \frac{dw}{2i\pi} \frac{\partial \psi(w)}{\partial m_{k, \tilde{f}}^f} S''(w). \quad (5.8)$$

The constraint (4.4) ensures that  $S(w)$  is single-valued along the contour  $\alpha$ . We can thus integrate by part twice in (5.8) to finally get

$$G_{f, \tilde{f}}^{\tilde{f}}(z) = \frac{1}{N} \sum_{k \geq 0} \frac{1}{z^{k+1}} \oint_{\alpha} \frac{dw}{2i\pi} \frac{\partial \psi''(w)}{\partial m_{k, \tilde{f}}^f} S(w). \quad (5.9)$$

Now, from (5.3) and using (1.7) and (1.9) we get

$$\frac{\partial \psi''(w)}{\partial m_{k, \tilde{f}}^f} = \frac{\partial \ln U(w)}{\partial m_{k, \tilde{f}}^f} = w^k (m(w)^{-1})_{\tilde{f}}^f. \quad (5.10)$$

The series over  $k$  in (5.9) can then be summed up easily and we find

$$G_{f, \tilde{f}}^{\tilde{f}}(z) = \frac{1}{2i\pi N} \oint_{\alpha} \frac{dw}{z-w} (m(w)^{-1})_{\tilde{f}}^f S(w). \quad (5.11)$$

The resulting contour integral can be computed by deforming the contour  $\alpha$  to infinity and picking the contributions from the poles at  $w = z$  and  $w = b_Q$ . This yields the formula (5.1).

## 6 Going on-shell and the anomaly equations

In this last section, we are going to solve the equations (2.6). Using (2.7), (2.8) and (2.9), we shall then be able to make the link between the off-shell generating functions (2.3), (2.4) and (2.5) that we have computed previously and the physical generating functions (1.14), (1.15) and (1.16). In particular, we are going to show that the latter satisfy the anomaly equations (1.17)–(1.20).

The starting point is the fundamental formula that relates the derivative of the quantum superpotential to the  $\beta_i$  contour integrals of  $S' dz$ ,

$$\frac{\partial W_{\text{mic}}}{\partial a_i} = -\frac{1}{N} \int_{\beta_i} S'(z) dz + W'(\mu_0) = 0. \quad (6.1)$$

This relation takes exactly the same form as in the theory with no flavor, equation (3.51) of [3]. The derivation given in this latter reference, which uses in particular the Riemann

bilinear relations, applies without modification to the present case. This perfect analogy is due to the fact that the poles, that are a priori present in the case with flavors, are eliminated when one takes derivatives with respect to  $a_i$ .

Let us first examine the consequences of (6.1) for the glueball operators. For arbitrary values of  $\mathbf{a}$ , we have seen in section 4 that  $S'(z; \mathbf{a})$  was well-defined on the curve  $\mathcal{C}$  (3.13). Denoting as usual with a hat the value on the second sheet, we deduce from (4.2) that

$$S'(z; \mathbf{a}) + \hat{S}'(z; \mathbf{a}) = NW''(z). \tag{6.2}$$

Integrating, we get

$$S(z; \mathbf{a}) + \hat{S}^{(i)}(z; \mathbf{a}) = N(W'(z) - W'(\mu_0)) + \int_{\beta_i} S'(z) dz, \tag{6.3}$$

where  $\hat{S}^{(i)}(z; \mathbf{a})$  denotes the analytic continuation of  $S$  through the cut  $I_i$ . For general values of  $\mathbf{a}$ ,  $S(z; \mathbf{a})$  is *not* defined on  $\mathcal{C}$ , since the analytic continuation through a branch cut depends on the particular branch cut that we choose. However, for the particular on-shell values  $\mathbf{a} = \mathbf{a}^*$ , the relation (6.1) is satisfied and thus the right-hand side of (6.3) no longer depends on  $i$ . The physical generating function (2.8) is thus well-defined on  $\mathcal{C}$ , with

$$\mathcal{S}(z) + \hat{\mathcal{S}}(z) = NW'(z). \tag{6.4}$$

A trivial calculation using this relation immediately implies that the combination  $A(z) = NW'(z)\mathcal{S}(z) - \mathcal{S}(z)^2$  has no branch cuts, i.e.  $\hat{A}(z) = A(z)$ . It cannot have poles from the discussion of section 4. Using the asymptotic condition  $S(z) = \mathcal{O}(1/z)$  at infinity, we conclude that it must be a polynomial. This implies the first anomaly equation (1.17).

We can proceed in exactly the same way to derive the other anomaly equations. It is straightforward to check that the left hand sides in (1.18), (1.19) and (1.20) have no branch cuts by using (6.4), (3.15) and (5.1) (these last two equations are valid for any  $\mathbf{a}$ , and thus in particular for  $\mathbf{a} = \mathbf{a}^*$ ). It is also straightforward to check that the residues of the possible poles all cancel by using the simple pole structure of the various generating functions that we have discussed in the previous sections. The asymptotics at infinity then implies that the right hand sides of (1.18), (1.19) and (1.20) must be polynomials.

Let us close this section with two remarks. First, we note that the anomaly polynomials in (1.17) and (1.20) can be obtained by acting on  $W_{\text{mic}}$  with first order differential operators  $J_n$  and  $L_n$  defined exactly as in equations (3.63) and (3.62) of ref. [3]. In particular, the Riemann bilinear relations used in [3] to make the derivations can be easily generalized to take into account the poles that appear in the generating functions  $R$  and  $G_f^{\tilde{f}}$  when flavors are present. On the other hand, to obtain (1.18) and (1.19) from variations of the microscopic quantum superpotential, one would have to include arbitrary boundary conditions for the quarks in the formalism and compute  $W_{\text{mic}}$  as a function of both  $\mathbf{a}$  and these quark boundary conditions. The anomaly polynomials in (1.18) and (1.19) would then follow by acting on  $W_{\text{mic}}$  with suitable first order differential operators containing partial derivatives with respect to the quark boundary conditions. In the present paper, we have preferred to integrate out the quarks exactly first and thus work with a microscopic superpotential that depends on  $\mathbf{a}$  only.

Our second remark concerns the set of solutions to the quantum equations of motion (2.6). We have shown that any solution must satisfy the anomaly equations on top of (3.22) which is valid off-shell. Conversely, the set of solutions to the anomaly equations that also satisfy (3.22) is known to be in one-to-one correspondence with the full set of quantum vacua of the theory (see for example [10–12] and references therein). One can show that all these solutions also automatically solve (2.6), with one rather trivial exception that is discussed below. A simple way to understand this point is as follows. First, a straightforward generalization of the analysis in [1] shows that vacua of any rank  $r \geq 1$  of the type  $|N_i, k_i; \nu_Q = 0\rangle$  are automatically included in the set of solutions. Second, one uses the fact that all the other vacua at the same rank can be obtained by analytic continuations [11, 12] and thus necessarily solve (2.6) as well.

There is an interesting point concerning the vacua having  $\nu_Q \neq 0$ . At the classical level, one has  $\nu_Q = 1$  when one of the  $a_i$  is equal to  $b_Q$ . At the quantum level, one might expect that the solutions to (2.6) associated with these vacua correspond also to having  $a_i = b_Q$ . This would be natural from the analysis in the appendix, that shows that if one imposes the boundary condition  $a_i = b_Q$ , then the quantum function  $R(z; \mathbf{a})$  has a pole at  $z = b_Q$  on the first sheet. However, what really happens depends on the cases one considers and can be more subtle. The subtlety comes from the fact that the variables  $\mathbf{a}$  can undergo non-trivial monodromies, as is well-known from the study of the moduli space in the  $\mathcal{N} = 2$  supersymmetric theories [15]. Due to these monodromies, the actual solution  $\mathbf{a} = \mathbf{a}^*$  to (2.6) corresponding to a vacuum with  $\nu_Q = 1$  can actually have all the  $a_i$ s different from the  $b_Q$ s.

The above discussion doesn't apply for the vacua of rank zero. These vacua have a completely broken gauge group and correspond to the cases where all the eigenvalues  $a_i$ s are equal to the  $b_Q$ s classically. From the discussion in the appendix, we know that the solution is trivial in these cases: the chiral operator expectation values do not get any quantum correction. Now, it turns out that these trivial solutions do not satisfy (2.6). The reason is that the procedure of integrating out the quarks become singular from the point of view of the microscopic quantum superpotential in these particular vacua. This can be easily illustrated since these vacua are purely classical. Integrating out the quarks from the tree-level superpotential

$$W_{\text{tree}} = \text{Tr } W(X) + {}^T \tilde{Q}^{\tilde{f}} m_{\tilde{f}}^f(X) Q_f \tag{6.5}$$

amounts to imposing the conditions

$$m_{\tilde{f}}^f(X) Q_f = 0 = {}^T \tilde{Q}^{\tilde{f}} m_{\tilde{f}}^f(X). \tag{6.6}$$

The resulting effective superpotential, obtained by plugging (6.6) into (6.5), is simply

$$W_{\text{mic}} = \text{Tr } W(X) = \sum_{i=1}^N W(a_i), \tag{6.7}$$

whose variations only yield  $W'(a_i) = 0$ . For these solutions, the matrix  $m_{\tilde{f}}^f(X)$  is invertible and  $Q_f = \tilde{Q}^{\tilde{f}} = 0$ . The superpotential (6.7) is thus missing the solutions for which  $m_{\tilde{f}}^f$



has zero eigenvalues and  ${}^T \tilde{Q}^f Q_f \neq 0$ . These solutions correspond precisely to the cases  $a_i = b_Q$ . When the rank of the solutions is  $r \geq 1$ , and contrary to the case  $r = 0$ , there are non-trivial quantum corrections and as we have explained above the solutions are actually obtained from (2.6).

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## A The special cases with $a_i = b_Q$

### A.1 A simple example: $N = N_f = 1$

Let us consider the solution for the generating function  $R(z; \mathbf{a})$  described in section 3 in the case  $N = N_f = 1$ ,  $U = z - b$  and  $V = 0$ . If we note  $a_1 = a$  and  $b_1 = b$ , we get

$$R(z) = \frac{1}{2(z-b)} + \frac{1}{\sqrt{(z-a+q)^2 - 4q(z-b)}} \left( 1 - \frac{z-a+q}{2(z-b)} \right) \quad (\text{A.1})$$

from (3.35) and (3.21). The function  $R(z)$  is two-valued, with asymptotics at infinity  $R(z) \sim 1/z$  on the first sheet and a pole at  $z = b$  on the second sheet. The branching points are given by the equation

$$(z-a+q)^2 - 4q(z-b) = (z-w^-)(z-w^+), \quad (\text{A.2})$$

which yields

$$w^\pm = a + q \pm 2\sqrt{q(a-b)}. \quad (\text{A.3})$$

When  $b \rightarrow a$ , we see that the two branching points collide and  $R$  reduces to

$$R(z) = \frac{1}{z-b}. \quad (\text{A.4})$$

As explained in the main text, this is a very general phenomenon: when  $b_Q \rightarrow a_i$ , the cut  $[w_i^-, w_i^+]$  closes and the pole at  $z = b_Q$  is on the first sheet. We are going to find this property again in the next subsection from a direct analysis of the sum over colored partitions.

### A.2 Direct analysis using the sums over colored partitions

Chiral correlators with fixed boundary conditions  $\mathbf{a}$  for the field  $X$  are given by a sum over colored partitions  $\vec{k}$  of the form

$$\langle \mathbf{a} | \mathcal{O} | \mathbf{a} \rangle = \lim_{\epsilon \rightarrow 0} \frac{\sum_{\vec{k}} \nu_{\vec{k}} \mathcal{O}_{\vec{k}}}{\sum_{\vec{k}} \nu_{\vec{k}}}, \quad (\text{A.5})$$

where the limit  $\epsilon \rightarrow 0$  corresponds to a vanishing  $\Omega$ -background. A colored partition is a collection  $\vec{k} = (k_1, \dots, k_N)$  of  $N$  ordinary partitions  $\mathbf{k}_i = \{k_{i,\alpha}\}$ ,  $k_{i,1} \geq k_{i,2} \geq \dots \geq k_{i,\tilde{k}_{i,1}} > 0$ . Much more details about these sums can be found for example in [2]. The measure over the set of colored partitions decomposes as

$$\nu_{\vec{k}} = q^{|\vec{k}|} \mu_{\vec{k}}^2(\mathbf{a}, \mathbf{t}, \epsilon) \mathcal{E}_{\vec{k}}(\mathbf{a}, \epsilon), \quad (\text{A.6})$$

where the dressing factor  $\mathcal{E}_{\vec{k}}$  gives the contribution from the integration over the quark fields,

$$\mathcal{E}_{\vec{k}} = \prod_{i=1}^N \prod_{\alpha=1}^{\tilde{k}_{i,1}} \prod_{\beta=1}^{k_{i,\alpha}} U(a_i + \epsilon(\beta - \alpha)). \quad (\text{A.7})$$

This formula generalizes the dressing factor obtained in [4] in the case of a linear mass function  $m_f^f(z)$ . The polynomial  $U$  is defined in (1.7). When  $\epsilon \rightarrow 0$ , the sums (A.5) are dominated by a single large colored partition described by the shape function  $f$  [13]. This has been used extensively in section 3.

The cases where some of the  $a_i$ s are equal to the  $b_Q$ s are special. What happens is very clear from the form of the dressing factor (A.7): if  $a_i = b_Q$ , only the trivial partition  $\mathbf{k}_i = \emptyset$  yields a non-zero contribution. In general, if  $p$  distinct  $a_i$ s are equal to  $p$  distinct  $b_Q$ s, the sum over colored partitions reduces to a sum over  $N - p$  ordinary partitions which can be computed as in section 3. The dominant colored partition is described by a smooth shape function  $f$  that extremizes a functional given by

$$\begin{aligned} \mathcal{F} = & -\frac{1}{8} \int dx dy (x - y)^2 (\ln|x - y| - 3/2) f''(x) f''(y) + \frac{1}{2} \int dx t(x) f''(x) \\ & + \sum_{Q=1}^{\ell} \frac{1 - 2\nu_Q}{4} \int dx (x - b_Q)^2 (\ln(x - b_Q) - 3/2) f''(x), \end{aligned} \quad (\text{A.8})$$

where  $\nu_Q = 1$  if  $b_Q = a_i$  and  $\nu_Q = 0$  otherwise. The constraints on  $f$  are similar to (3.4)–(3.7), except that now the support of  $f$  is made of  $N - p$  distinct intervals corresponding to the  $a_i$ s that are distinct from the  $b_Q$ s. One can solve this extremization problem as in section 3. The resulting generating function

$$R(z) = \sum_{Q=1}^{\ell} \frac{\nu_Q}{z - b_Q} + \frac{1}{2} \int_{\mathbb{R}} dy \frac{f''(y)}{z - y} \quad (\text{A.9})$$

is defined on a hyperelliptic curve of genus  $N - 1 - p$ , with  $p$  poles on the first sheet and  $N - p$  poles on the second sheet having residue one. As already emphasized, this is exactly the same solution as the one obtained starting from  $R(z; \mathbf{a})$  for generic values of  $\mathbf{a}$  and  $\mathbf{b}$  and then going to the special points  $a_i = b_Q$ .

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